Deformation of $\mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2)$-Modules of Symbols

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Abstract

We consider the $\mathfrak{sl}(2)$-module structure on the spaces of symbols of differential operators acting on the spaces of weighted densities. We compute the necessary and sufficient integrability conditions of a given infinitesimal deformation of this structure and we prove that any formal deformation is equivalent to its infinitesimal part. We study also the super analogue of this problem getting the same results.

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1 Introduction

Let $\mathfrak{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{R}$. Denote by $\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in \mathbb{R}[x] \right\}$ the space of polynomial weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ is a $\mathfrak{vect}(1)$-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(f dx^\lambda) = (gf + \lambda g' f)dx^\lambda.$$

Any differential operator A on $\mathbb{R}$ can be viewed as the linear mapping $f(dx)^\lambda \mapsto (Af)(dx)^\mu$ from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ (λ, μ in $\mathbb{R}$). Thus the space of differential operators is a $\mathfrak{vect}(1)$-module, denoted $\mathrm{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. The $\mathfrak{vect}(1)$ action is:

$$L^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X. \tag{1.1}$$

Each module $\mathrm{D}_{\lambda,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{S}_{\lambda,\mu} := gr\mathrm{D}_{\lambda,\mu}$ is called the *space of symbols*. The quotient-module $\mathrm{D}^k_{\lambda,\mu}/\mathrm{D}^{k-1}_{\lambda,\mu}$ is isomorphic to module of tensor densities $\mathcal{F}_{\lambda-\mu-k}$, the isomorphism is provided by the principal symbol σ_{pr} defined by

$$A = \sum_{i=0}^{k} a_i(x)\partial_x^i \mapsto \sigma_{pr}(A) = a_i(x)(dx)^{\mu-\lambda-k}$$

As a $\mathfrak{vect}(1)$-module, the space $\mathcal{S}_{\lambda,\mu}$ depends only on the difference $\delta = \mu - \lambda$, so that $\mathcal{S}_{\lambda,\mu}$ can be written as $\mathcal{S}_\delta$, and we have

$$\mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\delta-k}.$$

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Denote by D_δ the $\mathfrak{vect}(1)$-module of differential operators in $\mathcal{S}_\delta$.

The space $\mathrm{D}_{\lambda,\mu}$ cannot be isomorphic as a $\mathfrak{vect}(1)$-module to the corresponding space of symbols, but is a deformation of this space in the sense of Richardson-Neijenhuis [10].

If we restrict ourselves to the Lie subalgebra of $\mathfrak{vect}(1)$ generated by $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite dimensional $\mathfrak{sl}(2)$ modules, still denoted $\mathcal{F}_\lambda$, $\mathrm{D}_{\lambda,\mu}$ and D_δ.

We are also interested in the study of the analogue super structures, namely, we consider the superspace $\mathbb{R}^{1|1}$ with coordinates (x,θ) where θ is the odd variable: $\theta^2=0$. This superspace is equipped with the standard contact structure given by the distribution $\langle\overline{\eta}\rangle$ generated by the vector field $\overline{\eta}=\partial_\theta-\theta\partial_x$. That is, the distribution $\langle\overline{\eta}\rangle$ is the kernel of the following 1-form:

$$\alpha=dx+\theta d\theta.$$

Consider the superspace of polynomial functions

$$\mathbb{R}[x,\theta]=\{F(x,\theta)=f_0(x)+\theta f_1(x)\mid f_0,\,f_1\in\mathbb{R}[x]\}$$

and consider the superspace $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathbb{R}^{1|1}$. That is, $\mathcal{K}(1)$ is the superspace of polynomial vector fields on $\mathbb{R}^{1|1}$ preserving the distribution $\langle\overline{\eta}\rangle$:

$$\mathcal{K}(1)=\big\{X\in\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}^{1|1})\mid[X,\,\overline{\eta}]=F_X\overline{\eta}\quad\text{for some }F_X\in\mathbb{R}[x,\,\theta]\big\}.$$

The Lie superalgebra $\mathcal{K}(1)$ is spanned by the vector fields of the form:

$$X_F=F\partial_x-\frac{1}{2}(-1)^{p(F)}\overline{\eta}(F)\overline{\eta},\quad\text{where}\quad F\in\mathbb{R}[x,\theta].$$

We introduce the superspace $\mathfrak{F}_\lambda=\{F\alpha^\lambda\mid F\in\mathbb{R}[x,\theta]\}$ of λ-densities on $\mathbb{R}^{1|1}$. This space is a $\mathcal{K}(1)$-module for the action defined by

$$\mathfrak{L}^\lambda_{X_G}(F\alpha^\lambda)=(X_G+\lambda G')(F)\alpha^\lambda.$$

Similarly, we consider the $\mathcal{K}(1)$-module of linear differential operators, $\mathfrak{D}_{\nu,\mu}:=\mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\nu,\mathfrak{F}_\mu)$, which is the super analogue of the space $\mathrm{D}_{\nu,\mu}$. The $\mathcal{K}(1)$-action on $\mathfrak{D}_{\nu,\mu}$ is given by

$$\mathfrak{L}^{\lambda,\mu}_{X_F}(A)=\mathfrak{L}^\mu_{X_F}\circ A-(-1)^{p(A)p(F)}A\circ\mathfrak{L}^\lambda_{X_F}.\tag{1.2}$$

The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2)=\mathrm{Span}\left(X_1,\,X_\theta,\,X_x,\,X_{x\theta},\,X_{x^2}\right).$$

The space of even elements of $\mathfrak{osp}(1|2)$ is isomorphic to $\mathfrak{sl}(2)$:

$$(\mathfrak{osp}(1|2))_0=\mathrm{Span}(X_1,\,X_x,\,X_{x^2})=\mathfrak{sl}(2).$$

The super analogue of the space $\mathcal{S}_\delta$ is naturally the superspace (see [8]):

$$\mathfrak{S}_\delta=\bigoplus_{k\in\mathbb{N}}\mathfrak{F}_{\delta-\frac{k}{2}}.$$

Denote by $\mathfrak{D}_\delta$ the $\mathcal{K}(1)$-module of linear differential operators in $\mathfrak{S}_\delta$.

In this paper, we study the deformations of the structure of the $\mathfrak{sl}(2)$-modules $\mathcal{S}_\delta$ and their analogues the $\mathfrak{osp}(1|2)$-modules $\mathfrak{S}_\delta$. We exhibit the necessary and sufficient integrability conditions of a given infinitesimal deformation. We prove that any formal deformation is equivalent to its infinitesimal part and we give an example of deformation with one parameter.

2 Deformation

Deformation theory of Lie algebra homomorphisms was first considered with only one-parameter of deformation [7, 10, 14]. Recently, deformations of Lie (super)algebras with multi-parameters were intensively studied (see, e.g., [1, 2, 4, 5, 6, 11, 12, 13]).

Let $\rho_0 : \mathfrak{g} \to \mathrm{End}(V)$ be an action of a Lie (super)algebra $\mathfrak{g}$ on a vector (super)space V. It is well known that the first cohomology space $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$ determines and classifies infinitesimal deformations up to equivalence. Thus, if $\dim \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V)) = m$, then choose 1-cocycles $\Upsilon_1, \dots, \Upsilon_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$ and consider the infinitesimal deformation

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i \, \Upsilon_i,$$

where $t_1, \dots, t_m$ are independent parameters with $p(t_i) = p(\Upsilon_i)$. We try to extend this infinitesimal deformation to a formal one:

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i \, \Upsilon_i + \sum_{i,j} t_i t_j \, \rho_{ij}^{(2)} + \cdots ,$$

where $\rho_{ij}^{(2)}, \rho_{ijk}^{(3)}, \dots$ are linear maps from $\mathfrak{g}$ to $\mathrm{End}(V)$ with $p(\rho_{ij}^{(2)}) = p(t_i t_j), p(\rho_{ijk}^{(3)}) = p(t_i t_j t_k), \dots$ such that

$$[\rho(x), \rho(y) = \rho([x, y]), \quad x, \, y \in \mathfrak{g}. \tag{2.3}$$

All the obstructions become from the condition (2.3) and it is well known that they lie in $\mathrm{H}^2(\mathfrak{g}, \mathrm{End}(V))$. Thus, we will impose extra algebraic relations on the parameters $t_1, \dots, t_m$. Let $\mathcal{R}$ be an ideal in $\mathbb{C}[[t_1, \dots, t_m]]$ generated by some set of relations, the quotient

$$\mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R}$$

is a (super)commutative associative (super)algebra with unity, and we can speak about deformations with base $\mathcal{A}$, (see [4, 7] for details).

2.1 Deformation of the $\mathfrak{sl}(2)$-Modules of Symbols

Now we study the formal deformations of the $\mathfrak{sl}(2)$-module structure on the space of symbols:

$$\mathcal{S}_\delta = \bigoplus_{k \geq 0} \mathcal{F}_{\delta - k}.$$

The infinitesimal deformations are described by the cohomology space

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{D}_\delta) = \bigoplus_{i,j \geq 0} \mathrm{H}^1_{\mathrm{diff}} \, (\mathfrak{sl}(2), \mathrm{D}_{\delta - j, \delta - i})$$

where $\mathrm{H}^*_{\mathrm{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. In fact, Lecomte computed $\mathrm{H}^1_{\mathrm{diff}} \left(\mathfrak{sl}(2), \mathrm{D}_{\lambda, \lambda'}\right)$, see [9]. He showed that non-zero cohomology $\mathrm{H}^1_{\mathrm{diff}} \left(\mathfrak{vect}(1), \mathrm{D}_{\lambda, \lambda'}\right)$ only appear if $\lambda = \lambda'$ or $(\lambda, \lambda') = (\frac{1-k}{2}, \frac{1+k}{2})$ where $k \in \mathbb{N}^*$. Thus, we distinguish two cases:

(i) If $\delta \notin \frac{1}{2}(\mathbb{N}+2)$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{D}_\delta) = \bigoplus_{k \geq 0} \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{sl}(2), \mathrm{D}_{\delta-k,\delta-k}\right).$$

The space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{sl}(2), \mathrm{D}_{\lambda,\lambda}\right)$ is one dimensional and it is spanned by the cohomology classe of the cocycle A_λ given by

$$A_\lambda(F\tfrac{d}{dx})(f dx^\lambda) = F' f dx^\lambda.$$

(ii) If $2\delta = m \in (\mathbb{N}+2)$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathrm{D}_\delta) = \bigoplus_{k=[\frac{m+1}{2}]}^{m-1} \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{m-2k}{2}, \frac{2+2k-m}{2}}\right) \oplus \bigoplus_{k \geq 0} \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{m}{2}-k, \frac{m}{2}-k}\right).$$

The space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{m-2k}{2}, \frac{2+2k-m}{2}}\right)$ is two dimensional and spanned by the cohomology classes of the 1-cocycles, B_k and C_k given by

$$B_k(F\tfrac{d}{dx})(f dx^{\frac{m-2k}{2}}) = F' f^{(2k-m+1)} dx^{\frac{2+2k-m}{2}}, \; C_k(F\tfrac{d}{dx})(f dx^{\frac{m-2k}{2}}) = F'' f^{(2k-m)} dx^{\frac{2+2k-m}{2}}.$$

In our study, an infinitesimal deformation of the $\mathfrak{sl}(2)$-module structure on the space $\mathcal{S}_\delta$ is of the form

$$\mathcal{L}_X = L_X + \mathcal{L}^{(1)}_X, \tag{2.4}$$

where L_X is the Lie derivative of D_δ along the vector field X defined by (1.1), and

$$\mathcal{L}^{(1)}_X = \begin{cases} \sum_{k\geq 0} a_k\, A_{\frac{m}{2}-k}(X) & \text{if } \delta \notin \frac{1}{2}(\mathbb{N}+2) \\ \sum_{k\geq 0} a_k\, A_{\frac{m}{2}-k}(X) + \sum_{k=[\frac{m+1}{2}]}^{m-1} \left(b_k\, B_k(X) + c_k C_k(X)\right) & \text{if } 2\delta = m \in (\mathbb{N}+2), \end{cases}$$

and where a_k, b_k and c_k are independent parameters.

Theorem 2.1. *The following conditions are necessary and sufficient for integrability of the infinitesimal deformation (2.4):*

$$(2k-m+1) b_k a_{m-k-1} + c_k a_k - c_k a_{m-k-1} = 0, \quad [\tfrac{m+1}{2}] \leq k \leq m-1\ . \tag{2.5}$$

Moreover, any formal deformation is equivalent to its infinitesimal part.

Proof. Note that if $\delta \notin \frac{1}{2}(\mathbb{N}+2)$ then the parameters b_k and c_k can be assumed to be zero, and then, there are no integrability conditions. Assume that the infinitesimal deformation (2.4) can be integrated to a formal deformation

$$\mathcal{L}_X = L_X + \mathcal{L}^{(1)}_X + \mathcal{L}^{(2)}_X + \mathcal{L}^{(3)}_X + \cdots$$

where $\mathcal{L}^{(1)}_X$ is given by (2.1) and $\mathcal{L}^{(2)}_X$ is a quadratic polynomial in a_k, b_k and c_k with coefficients in $\mathcal{D}_\delta$. We compute the conditions for the second-order terms $\mathcal{L}^{(2)}$. Consider the quadratic terms of the homomorphism condition

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{2.6}$$

By a straightforward computation, the homomorphism condition (2.6) gives for the second-order terms the following equation

$$\delta(\mathcal{L}^{(2)}) = \frac{1}{2}\sum_{k=[\frac{m+1}{2}]}^{m-1}((2k-m+1)b_k a_{m-k-1} + c_k a_k - c_k a_{m-k-1})\Phi_{2k-m+1}, \qquad (2.7)$$

where Φ_k is the nontrivial 2 cocycle given by

$$\Phi_k(F\tfrac{d}{dx}, G\tfrac{d}{dx})(f dx^{\frac{1-k}{2}}) = (F^{'}G^{''} - F^{''}G^{'})f^{(k-1)}dx^{\frac{1+k}{2}} \qquad (2.8)$$

The condition (2.5) is necessary since the operator Φ_k is a nontrivial 2 cocycle spanning the space $\mathrm{H}^2_{\rm diff}\left(\mathfrak{sl}(2), \mathrm{D}_{\frac{m-2k}{2},\frac{2+2k-m}{2}}\right)$, (see [9]). The solution $\mathcal{L}^{(2)}$ of (2.7) can be chosen identically zero. Choosing the highest-order terms $\mathcal{L}^{(m)}$ with $m \geq 3$, also identically zero, one obviously obtains a deformation (which is of order 1 in a_k, b_k and c_k). □

Example 2.2. *Let us consider $\delta = \frac{m}{2} \in \frac{1}{2}(\mathbb{N}+2)$ and let $(\alpha_k)_{k\geq 0}$ be sequence of real numbers such that, for $[\frac{m+1}{2}] \leq k \leq m-1$, we have $\alpha_k \neq \alpha_{m-k-1}$. Put $b_k = t$, $a_k = \alpha_k t$ and $c_k = \frac{(2k-m+1)\alpha_k}{\alpha_k - \alpha_{m-k-1}}\, t$. So, we obtain a deformation of $\mathcal{S}_\delta$ with one parameter t:*

$$\mathcal{L} = L + t\sum_{k\geq 0}\alpha_k\, A_{\frac{m}{2}-k}(X) + t\sum_{k=[\frac{m+1}{2}]}^{m-1}\left(B_k(X) + \frac{(2k-m+1)\alpha_k}{\alpha_k - \alpha_{m-k-1}}\, C_k(X)\right).$$

Of course it is easy to give many other examples of true deformations with one parameter or with several parameters.

2.2 Deformation of the $\mathfrak{osp}(1|2)$-Modules of Symbols

We study the super analogous of the previous case. That is, we study deformations of the $\mathfrak{osp}(1|2)$-module of differential linear operators in the space of symbols on $\mathbb{R}^{1|1}$:

$$\mathfrak{S}_\delta = \bigoplus_{k\geq 0}\mathfrak{F}_{\delta-\frac{k}{2}}.$$

The infinitesimal deformations are described by the cohomology space

$$\mathrm{H}^1_{\rm diff}(\mathfrak{osp}(1|2), \mathfrak{D}_\delta) = \bigoplus_{i,j\geq 0}\mathrm{H}^1_{\rm diff}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\delta-\frac{j}{2},\delta-\frac{i}{2}}\right).$$

In [3], it was proved that non-zero cohomology $\mathrm{H}^1_{\rm diff}\left(\mathcal{K}(1), \mathfrak{D}_{\lambda,\lambda'}\right)$ only appear if $\lambda = \lambda'$ or $(\lambda,\lambda') = (\frac{1-k}{2}, \frac{k}{2})$ where $k \in \mathbb{N}$. Thus, as before, we have to distinguish two cases:

(i) If $\delta \notin \frac{1}{2}(\mathbb{N}+1)$, then

$$\mathrm{H}^1_{\rm diff}(\mathfrak{osp}(1|2), \mathfrak{D}_\delta) = \bigoplus_{k\geq 0}\mathrm{H}^1_{\rm diff}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\delta-\frac{k}{2},\delta-\frac{k}{2}}\right).$$

The space $\mathrm{H}^1_{\rm diff}\left(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{2\delta-k}{2},\frac{2\delta-k}{2}}\right)$ is one dimensional and it is spanned by the cohomology classe of the cocycle $\Upsilon'_{2\delta-k}$ given by

$$\Upsilon'_{2\delta-k}(F\tfrac{d}{dx}) = F'.$$

(ii) If $2\delta = m \in (\mathbb{N}+1)$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(1|2),\mathfrak{D}_\delta) = \bigoplus_{k=1}^{m} \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(1|2),\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right) \oplus \bigoplus_{k\geq 0} \mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(1|2),\mathfrak{D}_{\frac{m-k}{2},\frac{m-k}{2}}\right).$$

The space $\mathrm{H}^1_{\mathrm{diff}}\left(\mathfrak{osp}(1|2),\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)$ is two dimensional and spanned by the cohomology classes of the 1-cocycles, Υ_k and $\widetilde{\Upsilon}_k$ given by

$$\Upsilon_k(X_G) = (-1)^{|G|}\eta^2(G)\overline{\eta}^{2k-1}, \quad \widetilde{\Upsilon}_k(X_G) = (-1)^{|G|}((k-1)\eta^4(G)\overline{\eta}^{2k-3} + \eta^3(G)\overline{\eta}^{2k-2}).$$

Any infinitesimal deformation of the $\mathfrak{spo}(1|2)$-module structure on $\mathfrak{S}_\delta$ is of the form

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}^{(1)}_{X_F}, \tag{2.9}$$

where $\mathfrak{L}_{X_F}$ is the Lie derivative of $\mathfrak{D}_\delta$ along the vector field X_F defined by (1.2), and

$$\mathfrak{L}^{(1)}_{X_F} = \begin{cases} \sum_{k\geq 0} \mathfrak{a}_{2\delta-k}\, \Upsilon'_{2\delta-k}(X_F) & \text{if } \delta \notin (\frac{1}{2}\mathbb{N}+1) \\ \sum_{k\geq 0} \mathfrak{a}_{m-k}\, \Upsilon'_{m-k}(X_F) + \sum_{k=1}^{m} \left(\mathfrak{b}_k\, \Upsilon_k(X_F) + \mathfrak{c}_k \widetilde{\Upsilon}_k(X_F)\right) & \text{if } 2\delta = m \in (\mathbb{N}+1), \end{cases}$$

and where $\mathfrak{a}_k$, $\mathfrak{b}_k$ and $\mathfrak{c}_k$ are independent parameters.

Our main result in the super setting is the following

Theorem 2.3. *The following conditions are necessary and sufficient for integrability of the infinitesimal deformation (2.9):*

$$\mathfrak{b}_k\mathfrak{a}_{1-k} - \mathfrak{c}_k\mathfrak{a}_k + \mathfrak{c}_k\mathfrak{a}_{1-k} = 0, \quad 1 \leq k \leq m\ . \tag{2.10}$$

Moreover, any formal deformation is equivalent to its infinitesimal part.

Proof. Assume that the infinitesimal deformation (2.9) can be integrated to a formal deformation:

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}^{(1)}_{X_F} + \mathfrak{L}^{(2)}_{X_F} + \cdots$$

By a straightforward computation, the homomorphism condition

$$[\widetilde{\mathfrak{L}}_{X_F}, \widetilde{\mathfrak{L}}_{X_G}] = \widetilde{\mathfrak{L}}_{X_{\{F,G\}}}$$

gives for the second-order terms the following equation

$$\delta(\mathfrak{L}^{(2)}) = \sum_{k=1}^{m} (\mathfrak{b}_k\mathfrak{a}_{1-k} - \mathfrak{c}_k\mathfrak{a}_k + \mathfrak{c}_k\mathfrak{a}_{1-k})\Omega_k$$

where $\Omega_k : \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \to \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}$ is defined by

$$\Omega_k(X_F, X_G) = (-1)^{p(F)+p(G)}(k-1)(F'G''-F''G')\overline{\eta}^{2k-3} + (\overline{\eta}(F')G' - (-1)^{p(F)p(G)}F'\overline{\eta}(G'))\overline{\eta}^{2k-2}.$$

We ill prove the following lemma and then we conclude as for Theorem 2.1.

Lemma 2.4. *The map Ω_k is a nontrivial odd 2 cocycle.*

Proof. The map Ω_k is a 2 cocycle since it is the cup-product of 1 cocycles. It is easy to see that Ω_k is an odd map, so, $\Omega_k(\mathfrak{sl}(2)\times\mathfrak{sl}(2))\subset\left(\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)_1$. In [4] it was proved that, as $\mathfrak{sl}(2)$-module, we have

$$\left(\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}}\right)_1 \simeq \Pi\left(\mathrm{D}_{\frac{2-k}{2},\frac{k}{2}}\oplus \mathrm{D}_{\frac{1-k}{2},\frac{1+k}{2}}\right) \tag{2.11}$$

where Π is the change of parity operator. We check that the restriction of Ω_k to $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ is a nontrivial 2 cocycle. Indeed, let $X_F,\, X_G\in\mathfrak{sl}(2)\subset\mathfrak{osp}(1|2)$, it is easy to see that

$$(-1)^k\Omega_k(X_F,X_G)=(k-1)\Phi_{k-1}(X_F,X_G)\circ\partial_\theta-k\theta\Phi_k(X_F,X_G),$$

or equivalently, according to the decomposition (2.11), we have

$$(-1)^k\Omega_k|_{\mathfrak{sl}(2)\times\mathfrak{sl}(2)}=\Pi\circ((k-1)\Phi_{k-1}-k\Phi_k)$$

where Φ_k is the nontrivial 2 cocycle defined by (2.8). Thus, Ω is a nontrivial 2 cocycle. $\square$

Obviously, as for the $\mathfrak{sl}(2)$-module $\mathcal{S}_\delta$, it easy to construct many examples of true deformations of the $\mathfrak{osp}(1|2)$-module $\mathfrak{S}_\delta$ with one parameter or with several parameters.

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